

# Multi-Metric Gravity via Massive Gravity

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A generalization to the theory of massive gravity is presented which includes three dynamical metrics. It is shown that at the linear level, the theory predicts a massless spin-2 field which is decoupled from the other two gravitons which are massive and interacting. In this regime the matter should naturally couple to massless gravitons which introduce a preferred metric that is the average of the primary metrics. The cosmological solution of the theory shows the de-Sitter behavior with a function of mass as its cosmological constant. Surprisingly, it lacks any non-trivial solution when one of the metrics is taken to be Minkowskian and seems to enhance the predictions which suggest that there is no homogeneous, isotropic and flat solution for the standard massive cosmology.

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## I. INTRODUCTION AND SUMMARY

The general theory of relativity (GR) has been reigning supreme since its introduction by Einstein nearly a hundred years ago. The underlying principles upon which GR is founded, namely the principles of equivalence and general covariance, have given the theory, in spite of its inevitable shortcomings, an unprecedented breath and depth, enabling it to predict, solve and answer questions which seemed beyond the possibilities of any scientific endeavor over the past decades. That said, new observations and discoveries in the recent past seem to be beyond the predictive powers of the standard GR. The discovery of the accelerated expansion of the universe relating to dark energy and galaxy rotations curves relating to dark matter are some of the examples that necessitate modifications to the standard GR. Among many proposals to modify GR, Massive Gravity (MG), whose roots go back to the work of Fierz and Pauli [1] in 1939 has been gaining momentum in recent years after being brought back from oblivion in the 1970's where

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renewed interest in quantum field theory became pervasive. It is an effective field theory of gravity which, as the name suggests, is an extension of gravity with a massive spin-2 graviton. Its linear version coupled to a source was studied by van Dam, Veltman and Zakharov [2] who discovered that it makes predictions different from the linear GR even in the limit of the zero mass. This is known today as the vDVZ discontinuity. This problem is traced back to the degrees of freedom entering the theory, 5 for a massive and 2 for a massless spin-2 graviton. Attempts to remedy this shortcoming is suggested in [3]. However, Boulware and Deser [4] demonstrated that the theory suffers from the existence of ghosts, when non-linear self-interaction terms added. A fix was suggested in [5] where it was demonstrated that the resulting theory is unitary and ghost free. A comprehensive and well written review on this subject can be found in [6].

Recently, it has been proposed in [7] that massive gravity in a ghost free form can be written in a bimetric language. Both of these topics separately have had their supporters and followers. Bimetric theories are theories with two dynamical spin-2 fields. However, it has been shown that there are no consistent theories of interacting massless spin-2 fields. In general these bimetric theories suffer from having Boulware-Deser ghosts, and so are unstable [4]. The bimetric theory was first proposed by N. Rosen in [8] to give a tensor character to quantities like gravitational energy-momentum density pseudo-tensor. In this theory the tensor  $g_{\mu\nu}$  is regarded as a gravitational field that has no direct connection to geometry and so geometrization of gravity is given up [9]. Subsequently a large amount of work has been done to address cosmological problems in this context [10, 11].

Contrary to N. Rosen in [8] and [9], some of these works have kept the geometrical viewpoint of General Relativity. Milgrom has used two metrics for constructing relativistic formulation of MOND gravity [11] which naturally leads to MOND and dark energy effects. In this BIMOND theory, matter lives in the space-time described by one of the metrics which couples to another metric through an interaction term. Drummond in [10] introduced two vierbein bundles into the space-time manifold so that each bundle supports its own metric. One of this metrics is associated with matter and the other with gravity. This theory is a kind of modified theory of gravity that has the flexibility to permit the introduction of a length scale to explain the observed rotation curves of galaxies. On the other hand, in the last two years, the theory of massive gravity has opened a new era of research after the work of de Rham, Gabadadze and Tolley [12] where a new interaction term for a non-linear massive gravity was constructed, leading to a theory which becomes ghost free. The cosmology of such a model is considered and shown to be devoid of the flat FRW solutions [13]. However it has recently been shown that the open FRW solution does exist [14].

In ordinary formalism of massive theories of gravity an additional reference metric  $f_{\mu\nu}$  is required. It has been shown recently by S. F. Hassan and R. A. Rosen [7] that if we use the interaction term constructed in massive gravity theories, resulting in a ghost free massive theory, one will arrive at a bimetric theory of gravity which is free of the Boulware-Deser ghost. This model is important in the context of bimetric theories due to the form of definition of the potential term. The cosmology of such a bimetric theory is studied in [15, 16].

In this work we generalize the massive gravity scenarios to include three dynamical metrics. It will be clear that this trimetric formalism is very similar to  $N$ -metric models, though it has new predictive powers as in bimetric models. Naturally, by introducing  $N$  metrics,  $N$  gravitons are expected in its linear regime. This fact is similar to the Kaluza-Klein modes. In this context the existence of  $N$  gravitons and the interaction between them are considered to address an effective field theory for gravity [17, 18]. In [17] there is an interesting interpretation of massive gravitons using the Fierz-Pauli mass in that the authors assume to have  $N$  sites which are related by interaction through the mass terms. This interpretation can also be used in our assumption of  $N$  metrics, not in the Fierz-Pauli format but in the new ghost-free massive gravity formalism.

As mentioned above, at the linear level, we arrive at a theory which describes one massless spin-2 field which is decoupled from two interacting massive spin-2 fields. In addition, the cosmological solution of this model will be presented. This model, similar to the very recent results in bimetric cosmology [15, 16], has some de-Sitter solution with a cosmological constant which is a function of the masses. It is interesting that in this model which is more extensive than ordinary massive cosmology [13], there is no non-trivial cosmological solution if one assumes one of the metrics to be Minkowskian. So it seems that the result of [13] can be generalized to: “there is no non-trivial solution for massive cosmology (even in its multi-metric representation) by assuming one Minkowski metric”.

## II. THE MODEL

We start with a review of the non-linear massive gravity action presented in [12]. The non-linear ghost-free action which reduces to the Fierz-Pauli action at the linear limit is

$$S_{massive} = -M_{Pl}^2 \int d^4x \sqrt{-g} \left[ R - 2m^2 \sum_{n=0}^4 \beta_n e_n \left( \sqrt{g^{-1}f} \right) \right] \quad (1)$$

where the square root matrix is defined as  $\sqrt{A}\sqrt{A} = A$  for a general metric  $A$ . The  $e_k(\sqrt{g^{-1}f})$  are 5 elementary polynomials of the eigenvalues  $\lambda_n$  of the matrix  $\sqrt{g^{-1}f}$ . They are explicitly written as

$$\begin{aligned} e_0(\sqrt{g^{-1}f}) &= 1 \\ e_1(\sqrt{g^{-1}f}) &= \sum_{i=1}^4 \lambda_i \\ e_2(\sqrt{g^{-1}f}) &= \sum_{i<j} \lambda_i \lambda_j \\ e_3(\sqrt{g^{-1}f}) &= \sum_{i<j<k} \lambda_i \lambda_j \lambda_k \\ e_4(\sqrt{g^{-1}f}) &= \prod_{i=1}^4 \lambda_i = \det \sqrt{g^{-1}f} \end{aligned} \quad (2)$$

This implies that the highest order term in the interaction part of the action is simply a determinant of the background metric  $f_{\mu\nu}$ . We also note that one can write the polynomials  $e_k(\sqrt{g^{-1}f})$  in terms of trace of the matrix  $\sqrt{g^{-1}f}$ . One observes that the potential term in the action is symmetric under the transformation [7]

$$f \leftrightarrow g, \quad \beta_n \leftrightarrow \beta_{4-n}, \quad (3)$$

so one may consider the above interaction term to be the interaction for the metric  $g_{\mu\nu}$  with the background metric  $f_{\mu\nu}$  or the interaction for the metric  $f_{\mu\nu}$ . These observations led S. F. Hassan and R. A. Rosen [7] to add a kinetic term for the metric  $f_{\mu\nu}$  to the action and converted it to an action for a bimetric massive theory which is found to be ghost free. In the present work we generalize the work in [7] to become a non-linear ghost free action describing a trimetric model. The action then reads

$$\begin{aligned} S = & -M_g^2 \int d^4x \sqrt{-g} R_g - M_f^2 \int d^4x \sqrt{-f} R_f - M_h^2 \int d^4x \sqrt{-h} R_h \\ & + 2m_1^2 M_{gf}^2 \int d^4x \sqrt{-g} \sum_{n=0}^4 \beta_n e_n(\sqrt{g^{-1}f}) \\ & + 2m_2^2 M_{gh}^2 \int d^4x \sqrt{-g} \sum_{m=0}^4 \gamma_m e_m(\sqrt{g^{-1}h}) \\ & + 2m_3^2 M_{fh}^2 \int d^4x \sqrt{-f} \sum_{s=0}^4 \alpha_s e_s(\sqrt{f^{-1}h}) \end{aligned} \quad (4)$$

where  $R_g$ ,  $R_f$  and  $R_h$  are the Ricci scalars constructed by three metrics  $g_{\mu\nu}$ ,  $f_{\mu\nu}$  and  $h_{\mu\nu}$  respectively. We also have introduced three different Planck masses for each metric. We have for each interaction term, an effective Planck mass constructed by metrics included in the interaction

$$\begin{aligned} \frac{1}{M_{gf}^2} &\equiv \frac{1}{M_g^2} + \frac{1}{M_f^2} \\ \frac{1}{M_{gh}^2} &\equiv \frac{1}{M_g^2} + \frac{1}{M_h^2} \\ \frac{1}{M_{fh}^2} &\equiv \frac{1}{M_f^2} + \frac{1}{M_h^2}. \end{aligned} \quad (5)$$

### III. LINEAR THEORY

As is well known, bimetric theories in general, describe one massless and one massive spin-2 fields. In this section we will see that the above trimetric theory describes one massless spin-2 field and two massive spin-2 fields which in general interact with each other. We assume for simplicity, that the interaction terms are all given by the minimal model introduced in [7, 19]

$$\beta_0 = 3, \quad \beta_1 = -1, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 1, \quad (6)$$

and also for  $\gamma_n$  and  $\alpha_n$ . In this model the interaction terms can be written in terms of trace and determinant

$$2m_1^2 M_{gf}^2 \int d^4x \sqrt{-g} \left( 3 - \text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right), \quad (7)$$

and similarly for the other interactions.

Now we expand the metrics around the same fixed background[20]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{g}_{\mu\nu}, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{f}_{\mu\nu}, \quad h_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}. \quad (8)$$

To second order in perturbations, this trimetric minimal model reduces to the action

$$\begin{aligned} S = - \int d^4x & \left( M_g^2 \tilde{g}_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \tilde{g}_{\alpha\beta} + M_f^2 \tilde{f}_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \tilde{f}_{\alpha\beta} + M_h^2 \tilde{h}_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \tilde{h}_{\alpha\beta} \right) \\ & - \frac{1}{4} m_1^2 M_{gf}^2 \int d^4x \left[ \left( \tilde{g}_{\mu\nu}^{\mu} - \tilde{f}_{\mu\nu}^{\mu} \right)^2 - \left( \tilde{g}_{\mu}^{\mu} - \tilde{f}_{\mu}^{\mu} \right)^2 \right] \\ & - \frac{1}{4} m_2^2 M_{gh}^2 \int d^4x \left[ \left( \tilde{g}_{\mu\nu}^{\mu} - \tilde{h}_{\mu\nu}^{\mu} \right)^2 - \left( \tilde{g}_{\mu}^{\mu} - \tilde{h}_{\mu}^{\mu} \right)^2 \right] \\ & - \frac{1}{4} m_3^2 M_{fh}^2 \int d^4x \left[ \left( \tilde{f}_{\mu\nu}^{\mu} - \tilde{h}_{\mu\nu}^{\mu} \right)^2 - \left( \tilde{f}_{\mu}^{\mu} - \tilde{h}_{\mu}^{\mu} \right)^2 \right], \end{aligned} \quad (9)$$

where  $\mathcal{E}^{\mu\nu\alpha\beta}$  is the Einstein-Hilbert kinetic term and the background metric  $\bar{g}_{\mu\nu}$  is responsible to raise and lower the indices. Now by using the following transformations

$$\Upsilon'_{\mu\nu} \equiv \frac{1}{M^2} \left( M_g^2 \tilde{g}_{\mu\nu} + M_f^2 \tilde{f}_{\mu\nu} + M_h^2 \tilde{h}_{\mu\nu} \right) \quad (10)$$

$$\Phi_{\mu\nu} \equiv \tilde{g}_{\mu\nu} - \tilde{f}_{\mu\nu} \quad (11)$$

$$\Psi_{\mu\nu} \equiv \tilde{f}_{\mu\nu} - \tilde{h}_{\mu\nu} \quad (12)$$

$$\Omega_{\mu\nu} \equiv \tilde{g}_{\mu\nu} - \tilde{h}_{\mu\nu} \quad (13)$$

where  $M^2 \equiv M_g^2 + M_f^2 + M_h^2$  and obviously  $\Phi_{\mu\nu} + \Psi_{\mu\nu} + \Omega_{\mu\nu} = 0$ , one can transform the kinetic term in (9) to

$$M^2 \Upsilon'_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Upsilon'_{\alpha\beta} + \frac{M_g^2 M_f^2 M_h^2}{M^2 M_{fh}^2} \Phi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Phi_{\alpha\beta} + \frac{M_g^2 M_f^2 M_h^2}{M^2 M_{fg}^2} \Psi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Psi_{\alpha\beta} + \frac{M_g^2 M_h^2}{M^2} [\Phi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Psi_{\alpha\beta} + \Psi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Phi_{\alpha\beta}]$$

which is not diagonal due to the last two terms. Before making it diagonal let us use

$$\Phi'_{\mu\nu} \equiv \frac{M_g M_f M_h}{M M_{fh}} \Phi_{\mu\nu}, \quad \Psi'_{\mu\nu} \equiv \frac{M_g M_f M_h}{M M_{gf}} \Psi_{\mu\nu}$$

To make them diagonal one may now use the following transformations

$$\Phi''_{\mu\nu} \equiv \frac{1}{2} (\Phi'_{\mu\nu} + \Psi'_{\mu\nu}), \quad \Psi''_{\mu\nu} \equiv \frac{1}{2} (\Phi'_{\mu\nu} - \Psi'_{\mu\nu}),$$

to get

$$M^2 \Upsilon'_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Upsilon'_{\alpha\beta} + 2 \left( 1 + \frac{M_{fh} M_{gf}}{M_f^2} \right) \Phi''_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Phi''_{\alpha\beta} + 2 \left( 1 - \frac{M_{fh} M_{gf}}{M_f^2} \right) \Psi''_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Psi''_{\alpha\beta},$$

which is not yet canonical due to different weights. Eventually by assuming

$$\Upsilon_{\mu\nu} \equiv M \Upsilon'_{\mu\nu}, \quad \Pi_{\mu\nu} \equiv \sqrt{2 \left( 1 + \frac{M_{fh} M_{gf}}{M_f^2} \right)} \Phi''_{\mu\nu}, \quad \Xi_{\mu\nu} \equiv \sqrt{2 \left( 1 - \frac{M_{fh} M_{gf}}{M_f^2} \right)} \Psi''_{\mu\nu},$$

the kinetic term will be canonical as follows

$$\Upsilon_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Upsilon_{\alpha\beta} + \Pi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Pi_{\alpha\beta} + \Xi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Xi_{\alpha\beta},$$

with new fields  $\Upsilon$ ,  $\Pi$  and  $\Xi$ . Let us now use the above procedure for the potential term in Lagrangian (9). The corresponding potential term with a straightforward algebraic calculation is

$$-\frac{M_1^2}{4} (\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^\mu{}_\mu\Pi^\nu{}_\nu) - \frac{M_2^2}{4} (\Xi_{\mu\nu}\Xi^{\mu\nu} - \Xi^\mu{}_\mu\Xi^\nu{}_\nu) - \frac{\lambda}{4} (\Pi_{\mu\nu}\Xi^{\mu\nu} - \Pi^\mu{}_\mu\Xi^\nu{}_\nu),$$

where

$$\begin{aligned} M_1^2 &\equiv \frac{1}{2} \left( 1 + \frac{M_{fh}M_{gf}}{M_f^2} \right)^{-1} \frac{M^2}{M_g^2 M_f^2 M_h^2} \left[ M_{fh}^2 M_{gf}^2 (m_1^2 + m_3^2) + (M_{fh} + M_{gf})^2 (m_2^2 M_{gh}^2) \right] \\ M_2^2 &\equiv \frac{1}{2} \left( 1 - \frac{M_{fh}M_{gf}}{M_f^2} \right)^{-1} \frac{M^2}{M_g^2 M_f^2 M_h^2} \left[ M_{fh}^2 M_{gf}^2 (m_1^2 + m_3^2) + (M_{fh} - M_{gf})^2 (m_2^2 M_{gh}^2) \right] \\ \lambda &\equiv \left( 1 - \frac{M_{fh}^2 M_{fg}^2}{M_f^4} \right)^{-\frac{1}{2}} \frac{M^2}{M_g^2 M_f^2 M_h^2} \left[ M_{fh}^2 M_{gf}^2 (m_1^2 - m_3^2) + m_2^2 M_{gh}^2 (M_{fh}^2 - M_{gf}^2) \right]. \end{aligned}$$

The full Lagrangian for these new variables is therefore

$$\begin{aligned} \mathcal{L} &= \left( \Upsilon_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Upsilon_{\alpha\beta} + \Pi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Pi_{\alpha\beta} + \Xi_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \Xi_{\alpha\beta} \right) \\ &- \left[ \frac{M_1^2}{4} (\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^\mu{}_\mu\Pi^\nu{}_\nu) + \frac{M_2^2}{4} (\Xi_{\mu\nu}\Xi^{\mu\nu} - \Xi^\mu{}_\mu\Xi^\nu{}_\nu) + \frac{\lambda}{4} (\Pi_{\mu\nu}\Xi^{\mu\nu} - \Pi^\mu{}_\mu\Xi^\nu{}_\nu) \right]. \end{aligned} \quad (14)$$

It is obvious that one of the gravitons,  $\Upsilon$ , is massless though the other gravitons not only are massive but also interactive with arbitrary coefficients. This interaction term is a new prediction for having more than two gravitons and cannot be resolved by a new redefinition of the fields while taking the kinetic term canonical. This fact can be emphasized by noting that in the primary Lagrangian (9), three dimensional parameters exist. Consequently, these three dimensional parameters appear naturally after the field redefinitions. However, now two of them are represented as mass terms and the other as an interaction term. However, it is possible to resolve the interaction term by fine-tuning as “ $m_1^2 = m_3^2$  and  $M_h^2 = M_g^2$ ” which means that in the primary Lagrangian (9) there are just two dimensional parameters. An interesting point is that the massless graviton corresponds to

$$\Upsilon_{\mu\nu} \propto (M_g^2 \tilde{g}_{\mu\nu} + M_f^2 \tilde{f}_{\mu\nu} + M_h^2 \tilde{h}_{\mu\nu}),$$

which is exactly the average of the primary fields in (9). To have an idea on the masses, let us assume the special case of  $m^2 = m_1^2 = m_2^2 = m_3^2$  and  $\sqrt{3}M = M_g = M_f = M_h$  which means we have just one dimensional scale with the primary Lagrangian (4) being totally symmetric under changing the metrics. In this case the final gravitons have masses 0 and  $M_1^2 = M_2^2 = 3 \times \frac{m^2}{2}$  where  $\frac{m^2}{2}$  appears in (9) as mass term[21]. The interesting point is that the coefficient “3” is exactly the number of metrics which exist in the model. This argument can be generalized for  $N$ -metric formalism by a little linear algebra for matrices. So in  $N$ -metric formalism, one graviton is massless and other  $N - 1$  gravitons have masses as  $M_i^2 = N \times m^2, \forall i \in \{1, 2, \dots, N - 1\}$  where  $m^2$  is the mass before the field redefinitions.

It should be noticed that some of the above conclusion is true for the  $N$ -metric formalism. In fact in  $N$ -metric models with ghost-free potential terms such as those in (9), after the redefinition of fields, the model represents a massless graviton which is the average of the metrics and the  $N - 1$  massive gravitons with interaction. For a quick check let us innumerate the number of dimensional parameters. For the primary Lagrangian the number of interaction terms are  $\binom{N}{2}$ , but after field redefinition one field becomes massless. The other  $N - 1$  remaining fields are massive and present  $N - 1$  dimensional parameters as well as  $\binom{N-1}{2}$  new interactive terms. So finally we have  $N - 1 + \binom{N-1}{2}$  dimensional parameters which is exactly  $\binom{N}{2}$ . In the language of [17], this model can be understood as follows; the primary Lagrangian (4) represents  $N$  sites which are linked to each other with  $\binom{N}{2}$  links. This representation can be reduced to a representation with  $N$  sites with *one* site which has no link to the remaining  $N - 1$  sites. But these  $N - 1$  sites are linked not only to each other but also have self-links or respectively have interactions and masses.

### A. Coupling to matter

As mentioned in [7], the coupling of multi-metric theories in the context of massive gravity to matter fields has not yet been solved. However, by considering the symmetry it seems as if one can constrain the coupling of the matter

to gravitons. By looking at (9) and assuming that there is no preference between the primary metrics one may then conclude that each kind of coupling is natural to be symmetric under the interchange of the metrics. In addition, from the linearized theory, it is obvious that there is a natural choice for the symmetric form of the metric which can be constructed out of their average. So, just by considering the symmetry, it seems that any correct coupling of matter should be to a symmetric combination of metrics which reduces to their average in the linear form. This argument becomes more significant if we note that the average of the metrics in the linear theory is simply the massless graviton. This means that whenever the linear theory is applicable, the matter couples to massless gravitons directly and does not see the massive ones. The existence of massive gravitons may show themselves in higher order terms which in turn may be verified in the context of cosmological perturbations. In the power spectrum of cosmological perturbations (e.g. curvature perturbation) there is no difference between Einstein gravity and massive gravity. However, in the bispectrum of perturbations (i.e. non-Gaussianity) the massive gravity should have different predictions.

## IV. COSMOLOGICAL SOLUTIONS

### A. Cosmological equations

In this section we obtain the cosmological solutions of the above trimetric theory. In the massive gravity case, it has been shown that the flat FRW space-time is not the solution of the model [13]. However, open FRW solutions do exist [14]. Recently, the cosmological solutions for the non-linear ghost-free bimetric action was obtained in [15, 16]. Now we generalize them to the trimetric case.

Let us assume that all three metrics can be described in terms of an isotropic and homogeneous space-times

$$\begin{aligned} ds_g^2 &= -N(t)^2 dt^2 + a(t)^2 (dr^2 + r^2 d\Omega^2) \\ ds_f^2 &= -M(t)^2 dt^2 + b(t)^2 (dr^2 + r^2 d\Omega^2) \\ ds_h^2 &= -Q(t)^2 dt^2 + w(t)^2 (dr^2 + r^2 d\Omega^2). \end{aligned} \quad (15)$$

Plugging these into action (4), we can read the reduced Lagrangian as follows

$$\begin{aligned} \mathcal{L}_{red} &= 6 \left( M_g^2 \frac{a\dot{a}^2}{N} + M_f^2 \frac{b\dot{b}^2}{M} + M_h^2 \frac{w\dot{w}^2}{Q} \right) \\ &+ 2m_1^2 M_{gf}^2 \left( \beta_0 N a^3 + \beta_1 (M a^3 + 3N a^2 b) + 3\beta_2 ab(Ma + Nb) + \beta_3 b^2(3Ma + Nb) + \beta_4 M b^3 \right) \\ &+ 2m_2^2 M_{gh}^2 \left( \gamma_0 N a^3 + \gamma_1 (Q a^3 + 3N a^2 w) + 3\gamma_2 aw(Qa + Nw) + \gamma_3 w^2(3Qa + Nw) + \gamma_4 Q w^3 \right) \\ &+ 2m_3^2 M_{fh}^2 \left( \alpha_0 M b^3 + \alpha_1 (Q b^3 + 3M b^2 w) + 3\alpha_2 bw(Qb + Mw) + \alpha_3 w^2(3Qb + Mw) + \alpha_4 Q w^3 \right). \end{aligned} \quad (16)$$

The equations of motion can be obtained by varying the reduced Lagrangian with respect to scale factors with the result

$$\begin{aligned} 6M_g^2 \left( \frac{\dot{a}^2}{N} + 2\frac{a\ddot{a}}{N} - 2\frac{a\dot{a}\dot{N}}{N^2} \right) \\ - 2m_1^2 M_{gf}^2 \left( 3\beta_0 N a^2 + 3\beta_1 a(Ma + 2Nb) + 3\beta_2 b(2Ma + Nb) + 3\beta_3 M b^2 \right) \\ - 2m_2^2 M_{gh}^2 \left( 3\gamma_0 N a^2 + 3\gamma_1 a(Qa + 2Nw) + 3\gamma_2 w(2Qa + Nw) + 3\gamma_3 Q w^2 \right) = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} 6M_f^2 \left( \frac{\dot{b}^2}{M} + 2\frac{b\ddot{b}}{M} - 2\frac{b\dot{b}\dot{M}}{M^2} \right) \\ - 2m_1^2 M_{gf}^2 \left( 3\beta_1 N a^2 + 3\beta_2 a(Ma + 2Nb) + 3\beta_3 b(2Ma + Nb) + 3\beta_4 M b^2 \right) \\ - 2m_3^2 M_{fh}^2 \left( 3\alpha_0 M b^2 + 3\alpha_1 b(Qb + 2Mw) + 3\alpha_2 w(2Qb + Mw) + 3\alpha_3 Q w^2 \right) = 0, \end{aligned} \quad (18)$$

$$\begin{aligned}
& 6M_h^2 \left( \frac{\dot{w}^2}{Q} + 2\frac{w\ddot{w}}{Q} - 2\frac{w\dot{w}\dot{Q}}{Q^2} \right) \\
& - 2m_2^2 M_{gh}^2 \left( 3\gamma_1 N a^2 + 3\gamma_2 a(Qa + 2Nw) + 3\gamma_3 w(2Qa + Nw) + 3\gamma_4 Qw^2 \right) \\
& - 2m_3^2 M_{fh}^2 \left( 3\alpha_1 M b^2 + 3\alpha_2 b(Qb + 2Mw) + 3\alpha_3 w(2Qb + Mw) + 3\alpha_4 Qw^2 \right) = 0,
\end{aligned} \tag{19}$$

and variation of the Lagrangian with respect to the lapse functions become

$$\begin{aligned}
& 6M_g^2 \frac{a\dot{a}^2}{N^2} - 2m_1^2 M_{gf}^2 \left( \beta_0 a^3 + 3\beta_1 b a^2 + 3\beta_2 a b^2 + \beta_3 b^3 \right) \\
& - 2m_2^2 M_{gh}^2 \left( \gamma_0 a^3 + 3\gamma_1 w a^2 + 3\gamma_2 a w^2 + \gamma_3 w^3 \right) = 0,
\end{aligned} \tag{20}$$

$$\begin{aligned}
& 6M_f^2 \frac{b\dot{b}^2}{M^2} - 2m_1^2 M_{gf}^2 \left( \beta_1 a^3 + 3\beta_2 b a^2 + 3\beta_3 a b^2 + \beta_4 b^3 \right) \\
& - 2m_3^2 M_{fh}^2 \left( \alpha_0 b^3 + 3\alpha_1 w b^2 + 3\alpha_2 b w^2 + \alpha_3 w^3 \right) = 0,
\end{aligned} \tag{21}$$

$$\begin{aligned}
& 6M_h^2 \frac{w\dot{w}^2}{Q^2} - 2m_2^2 M_{gh}^2 \left( \gamma_1 a^3 + 3\gamma_2 w a^2 + 3\gamma_3 a w^2 + \gamma_4 w^3 \right) \\
& - 2m_3^2 M_{fh}^2 \left( \alpha_1 b^3 + 3\alpha_2 w b^2 + 3\alpha_3 b w^2 + \alpha_4 w^3 \right) = 0.
\end{aligned} \tag{22}$$

## B. Solution I

In this section we are going to show that there is a solution for the above equations. To do this we suppose

$$a = b = w \tag{23}$$

and  $N = M = Q = 1$ , that is, we work in the comoving gauge. Equation (20) reduces to two disjoint [22] equations

$$\begin{aligned}
& \dot{a} = \pm \omega a, \\
& \omega \equiv \frac{1}{\sqrt{3}M_g} \left[ m_1^2 M_{gf}^2 \left( \beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 \right) + m_2^2 M_{gh}^2 \left( \gamma_0 + 3\gamma_1 + 3\gamma_2 + \gamma_3 \right) \right]^{\frac{1}{2}}.
\end{aligned} \tag{24}$$

Obviously, the solutions are  $a = e^{-\omega t}$  and  $a = e^{+\omega t}$  for minus and plus signs respectively. The latter, which is more interesting, is a de-Sitter solution with a cosmological constant  $\Lambda = 3\omega^2$ . Now it should be checked if this solution also satisfies the first equation in (17) for  $a = b$  in the comoving gauge. This equation is

$$\dot{a}^2 + 2a\ddot{a} = 3\omega^2 a^2, \tag{25}$$

where  $\omega$  has been defined previously. Obviously, both of  $a = e^{-\omega t}$  and  $a = e^{+\omega t}$  satisfy the above equation. Now we should check what are the conditions imposed on the remaining equations. One can simply find that the following equation should be satisfied due to (21) and (22)

$$\begin{aligned}
\omega & \equiv \frac{1}{\sqrt{3}M_g} \left[ m_1^2 M_{gf}^2 \left( \beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 \right) + m_2^2 M_{gh}^2 \left( \gamma_0 + 3\gamma_1 + 3\gamma_2 + \gamma_3 \right) \right]^{\frac{1}{2}} \\
& = \frac{1}{\sqrt{3}M_f} \left[ m_1^2 M_{gf}^2 \left( \beta_1 + 3\beta_2 + 3\beta_3 + \beta_4 \right) + m_3^2 M_{fh}^2 \left( \alpha_0 + 3\alpha_1 + 3\alpha_2 + \alpha_3 \right) \right]^{\frac{1}{2}} \\
& = \frac{1}{\sqrt{3}M_h} \left[ m_2^2 M_{gh}^2 \left( \gamma_1 + 3\gamma_2 + 3\gamma_3 + \gamma_4 \right) + m_3^2 M_{fh}^2 \left( \alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4 \right) \right]^{\frac{1}{2}},
\end{aligned}$$

where the first line is the definition of  $\omega$  and the last lines should be seen as constraints on the parameters such that  $a = b = w = e^{\pm\omega t}$ . It is obvious that equations (18) and (19) are automatically satisfied as equation (17).

### C. Solution II

Let us now see if in such a massive cosmology with more than two metrics it is possible to have a solution with one Minkowski metric. This solution is impossible for the bimetric case[23]. This assumption may be interesting because of its resemblance to ordinary massive cosmology [13] where a static metric together with a dynamical one exist.

Using equation (22) and assuming that  $w$  is a constant, one gets the following relation

$$m_2^2 M_{gh}^2 \left( \gamma_1 a^3 + 3\gamma_2 w a^2 + 3\gamma_3 a w^2 + \gamma_4 w^3 \right) = -m_3^2 M_{fh}^2 \left( \alpha_1 b^3 + 3\alpha_2 w b^2 + 3\alpha_3 b w^2 + \alpha_4 w^3 \right)$$

which should be satisfied for all  $a$  and  $b$ . Since  $a$  and  $b$  are dynamical fields, the above condition is satisfied if and only if RHS and LHS are equal term by term. The general solution for this is as follows [24]

$$a = b, \quad w = 1, \quad m_2^2 M_{gh}^2 \gamma_i = -m_3^2 M_{fh}^2 \alpha_i \quad i \in \{1, 2, 3, 4\}. \quad (26)$$

What is done here represents a possible cosmological solution for the trimetric formalism of massive gravity. This solution is deduced by assuming that one of the metrics is static. This property makes this solution interesting and comparable to the assumptions in [13]. Now, equation (20) with the above assumption becomes

$$6M_g^2 a \dot{a}^2 = 2m_1^2 M_{gf}^2 \left( \beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 \right) a^3 + 2m_2^2 M_{gh}^2 \left( \gamma_0 a^3 + 3\gamma_1 a^2 + 3\gamma_2 a + \gamma_3 \right), \quad (27)$$

and equation (21) with  $a = b$

$$6M_f^2 a \dot{a}^2 = 2m_1^2 M_{gf}^2 \left( \beta_1 + 3\beta_2 + 3\beta_3 + \beta_4 \right) a^3 + 2m_3^2 M_{fh}^2 \left( \alpha_0 a^3 + 3\alpha_1 a^2 + 3\alpha_2 a + \alpha_3 \right). \quad (28)$$

By comparing the above two equations and our assumption of taking one of the metrics as static (26), it is easy to see that the following conditions have to be satisfied by the parameters

$$\frac{1}{M_g^2} \left[ m_1^2 M_{gf}^2 (\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3) + m_2^2 M_{gh}^2 \gamma_0 \right] = \frac{1}{M_f^2} \left[ m_1^2 M_{gf}^2 (\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4) + m_3^2 M_{fh}^2 \alpha_0 \right]$$

$$\gamma_i = \alpha_i = 0, \quad i \in \{1, 2, 3\}.$$

Actually, with the above conditions one should consider the remaining equations to get a correct solution. However, there is no need to pursue the matter any further in this case since, as we saw, the necessary condition of taking one of the metrics static is the same as assuming  $\gamma_i = \alpha_i = 0$  for  $i \in \{1, 2, 3\}$ . By looking at Lagrangian (16), it is easy to find that this condition prevents the third metric in (15) interacting with others. So this solution is the trivial one and is without any importance.

Finally, we showed that as a result of [13], there is no non-trivial solution for the massive cosmology studied here if, at least, one of the metrics is static.

### V. CONCLUSIONS AND QUESTIONS

We have considered the generalization of the work done in [7], from a bimetric theory to a trimetric theory. This trimetric theory in fact has many of the consequences of the  $N$ -metric theory, with the advantage of simplicity. What we get in this paper, is the conclusion that, in the multi-metric theories, only one graviton becomes massless at the linear level, which is the average of the  $N$  metrics. The other  $N - 1$  spin-2 fields become massive and in general interact with each other. This has a general conclusion that ordinary matter must couple to metrics in such a way that at the linear level only the coupling to the average metric survives. This constraint on the form of the metric may be important when one addresses the subtleties of the multi-metric formalism. In this formalism, the geometrical interpretation of metrics e.g. the meaning of the distance and covariant derivative (parallel transportation) are yet unresolved. But with the above restriction which comes from coupling to matter a large class of multi-metric formulations are ruled out.

The cosmology of this multi-metric theory is also interesting and generalizes the result in massive cosmology [13]. We have seen in this paper that, if we take one of the metrics as Minkowskian, it will become impossible to get a non-trivial cosmological solution for the theory. This result is however true only in a flat case. It is a matter of further investigation to see if the theory can have non-trivial non-flat cosmological solutions, as in the massive cosmology



case [14]. It is also crucial to consider the cosmological perturbation for FRW cosmology in this context due to the existence of a lot of observational data. From a theoretical viewpoint, when having more than one metric, it is natural to get not only the adiabatic mode but also entropy perturbations without any need for extra matter fields.

We should also mention that the multi-metric theory constructed here is not proven to be ghost-free [25]. In fact, it is possible that the interaction terms introduced here may not suffice to cancel all Boulware-Deser ghosts of all spin-2 fields. This would require further investigation in a future work.

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  - [20] Note that our definition of perturbations is different with [7] due to a factor. It does not affect the final results.
  - [21] Note that the dividing by "2" is a consequence of our assumptions for this special case i.e.  $m^2 = m_1^2 = m_2^2 = m_3^2$  and  $\sqrt{3}M = M_g = M_f = M_h$  and definitions (5)
  - [22] The linear combination of their solutions is not a solution.
  - [23] This is obvious from recent works on this topic [15] and [16] though they did not mention it. The procedure is similar to what we have done for trimetric model.

- [24] One can assume  $a = \zeta b$  and  $w = c$  for constant  $\zeta$  and  $c$ . But these assumptions will reduce to (26) by a re-definition of  $\alpha_i$  and  $\gamma_i \forall i \in \{1, 2, 3, 4\}$  without any physical consequences.
- [25] However it seems the mechanism for bimetric model (see the second reference in [7]) may be applicable.